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AMPLITUDE DISPERSION AND STABILITY OF DISSIPATIVE WEAKLY NON-LINEAR WAVES

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ABSTRACT

A perturbation method to solve non-linear dissipative wave equations using ideas of Krylov and Bogolyubov is presented. The method is compared to Whitham's theory. Dispersion relations for non-linear dissipative waves, including amplitude dispersion, are discussed, Furthermore, stability problems of such waves are investigated.

INTRODUCTION

Mathematical progress during the last years makes it possible to investigate the dispersion of non-linear wave equations. Non-linear oscillators and waves are important models in hydrodynamics and in plasma physics. By the term "non-linear" we want to express that the ordinary differential equation describing the propagation of the waves are assumed to be non-linear equations. The terms "dispersion" and "weakly non-linear" will be discussed later.

H. Lashingsky[1] has discussed in detail the motivation to investigate mathematical models for non-linear modes in plasmas and he also presented such a model[2] starting from a non-linear oscillator equation

$$\ddot{x} + \omega^2 x = - \varepsilon F(x, \dot{x}) \tag{1}$$

To discuss the solutions of this equation, Lashinsky used the method of averaging in conjunction with the technique of variation of parameters (Krylov-Bogolyubov method[3]). For $\varepsilon=0$, Equation (1) has the

solution

$$x = A \cos(\omega t + \psi) \tag{2}$$

where now $A \to A(t)$, $\psi \to \psi(t)$ for $\varepsilon \neq 0$. Under the assumptions $\varepsilon << 1$, $\dot{A}/A << \omega$, $\dot{\psi}/\psi << \omega$, which express a weak non-linearity, by the so-called method of averaging (over one period) the leading terms in a Fourier expansion can be found and finally

$$< \dot{A} > \; = - \frac{\varepsilon}{2\pi\omega} \int_0^{2\pi} F(A \cos\theta - \omega A \sin\theta) \sin\theta \, d\theta \tag{3}$$

where $\Theta = \omega t + \psi$

The right hand sides are functions of the amplitude A only since the time was averaged out over one period. From (3), the amplitude A(t) as a function of time t can be computed[2]. Lashinsky also considered non-linear waves in bounded plasmas. He investigated the equation

$$\frac{1}{C^2} \phi_{tt} - \nabla^2 \phi = - \varepsilon F(\phi, \phi_t) \tag{4}$$

by using expansions of the type

$$\phi = \sum_\lambda a_\lambda(t) \; \phi_\lambda(x) \tag{5}$$

The orthogonality conditions of the ϕ_λ and the period averaging process result in similar equations like (3).

The averaging over the period was also used by Tam[4]. He considers the propagation of non-linear dispersive waves in a cold non-dissipative plasma. He uses a perturbation technique and introduced fast and slow variables giving the periodic and the non-linear (averaged) part of the solution. Also Luke[5] uses this method in his investigation of the non-linear wave equation

$$\frac{1}{c^2}\,\phi_{tt} - \nabla^2\phi = F(\phi) \tag{6}$$

He demonstrates that his perturbation technique is equivalent to Whitham's averaged Lagrangian method.[6, 7, 8]

In this paper we extend Whitham's theory of the averaged Lagrangian to dissipative waves. Tam[16] points out that the effect of collisional dissipation on non-linear dispersion is important. Collisional damping may reduce the effect of non-linear instability.

DISPERSION RELATION AND AVERAGED LAGRANGIAN

We consider waves $\phi(\vec{x}, t)$ which satisfy a non-linear partial differential equation of such a form that the linear wave equation can be split off. Rotating our coordinate system so that the wave vector $\vec{k}$ points into the direction of the x-axis we write for our wave equation

$$\frac{1}{c^2}\,\phi_{tt} - \phi_{xx} + b\phi + g\phi_t = -V'(\phi) + N(\phi_t) + \phi_t G(\phi) \tag{7}$$

where V', N and G are non-linear functions, $V' = \frac{dv}{d\phi}$, c, b and g are constants which may depend on ω. We now define a phase surface

$$\Theta(x, t) = \text{const} \tag{8}$$

which has the property that all points (x, t) on it have the same value of the wave function ϕ. From (8) we have

$$d\Theta = \Theta_x d_x + \Theta_t dt = 0 \tag{9}$$

so that points moving with the speed

$$\frac{dx}{dt} = \frac{-\Theta_t}{\Theta_x} \tag{10}$$

see a constant phase Θ. Defining wave number and frequency by

$$\Theta_x = k, \quad -\Theta_t = \omega \tag{11}$$

we see that (10) is the phase speed. In the three dimensional case we have $\nabla\Theta = \vec{k}$, and therefore

$$\text{curl } \vec{k} = 0 \tag{12}$$

which indicates that wave crests are neither vanishing nor splitting into two or more crests.[9] From (11) and (12) the <u>conservation equation of wave crests</u>, Equation (13), follows:

$$\frac{\partial \vec{k}}{\partial t} + \nabla\omega = 0 \tag{13}$$

In a similar way it can be shown [9, 10] that a point moving with the group velocity

$$\left(\frac{dx}{dt}\right)_g = \frac{d\omega}{dk} \tag{14}$$

sees ω unchanged. After this disgression, we return to Equation (7). Its Lagrangian reads

$$L(\phi, \phi_x, \phi_t) = \frac{1}{c^2} \; \frac{1}{2}\,\phi_t^{\;2} - \frac{1}{2}\,\phi_x^{\;2} - \frac{b}{2}\,\phi^2 - V(\phi) \tag{15}$$

Since the system is dissipative, the Euler Lagrange equations read[11, 12]

$$\frac{\partial}{\partial x}\frac{\partial L}{\partial \phi_x} + \frac{\partial}{\partial t}\frac{\partial L}{\partial \phi_t} - \frac{\partial L}{\partial \phi} = -\,g\phi_t + N(\phi_t) + \phi_t\,G(\phi) \tag{16}$$

where $N(\phi_t)$ may be of the form

$$N(\phi_t) = -\,\varepsilon\phi_t^{\,2n-1} \tag{17}$$

where n = 1 1/2, 2, 2 1/2 ... Substituting the Lagrangian (15) into (16) we immediately obtain our wave equation (7). We will now make a classification of our wave equation:

I. $N = 0$, $V' = 0$: The equation is linear; k and ω are independent of x and t.

1. $b = 0$, $g = 0$: No dispersion, no dissipation.

$$\phi = Ae^{ikx-i\omega t}, \quad \omega = \pm ck \tag{18}$$

2. $b \neq 0$, $g = 0$: Frequency dispersion, no dissipation.

$$\phi = Ae^{ikx-i\omega t}, \quad \omega = \pm c\sqrt{k^2+b^2} \tag{19}$$

Any function $\omega(k)$, with exception of the definition (18), is called a <u>dispersion relation of a linear wave equation</u>. Such dispersion relations may also be found by a Laplace-Fourier transformation of the linear wave equation.[13]

3. $b = 0$, $g \neq 0$: Dissipative Case.

$$\phi = Ae^{ikx-i\omega t}, \quad \omega = -\frac{igc^2}{2} \pm c\sqrt{k^2\phi - \frac{g^2c^2}{4}} \tag{20}$$

4. $b \neq 0$, $g \neq 0$: Dissipative case, ϕ like in (20).

$$G(\omega, k) \equiv \frac{\omega^2}{c^2} - k^2 - b + gi\omega = 0 \tag{21}$$

II. If neither N, G nor V vanish, the wave equation is non-linear and solutions cannot be given (except in special cases[5]). If, however, the non-linearity is <u>weak</u>, which we define by

$$< \Theta_{xx} > = \frac{1}{2\pi}\int_0^{2\pi} \Theta_{xx}d\Theta = 0, \quad < \Theta_{tt} > = \frac{1}{2\pi}\int_0^{2\pi} \Theta_{tt}d\Theta = 0 \tag{22}$$

so that $\Theta_x = k$ and $\Theta_t = -\omega$ become constant "in the average," i.e. when appearing under an integral $\int_0^{2\pi} \ldots d\Theta$, then a dispersion relation of a non-linear wave equation can be derived. It turns out, that ω is not only a function of k (frequency dispersion) but also of the amplitude A (amplitude dispersion).

In the dissipationless case $(g = G = N = 0)$, Whitham suggested[6,7,15] the existence of a variational principle

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \Theta_t} + \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial \Theta_x} = 0 \tag{23}$$

for the averaged Lagrangian

$$\mathcal{L} = \frac{1}{2\pi}\int_0^{2\pi} L(\Theta, \Theta_x, \Theta_t, A)d\Theta = \mathcal{L}(\Theta_x, \Theta_t, A) \tag{24}$$

and he assumed that the variation of $\mathcal{L}$ with respect to the amplitude A or to the energy $E = A^2/2$, i.e.

$$\mathcal{L}_E = 0 \quad \text{or} \quad \mathcal{L}_A = 0 \tag{25}$$

gives the dispersion relation of the non-linear equation. Equations (13), (23) and (25) determine the 3 functions $k(x, t)$ $\omega(x, t)$ and $A(x, t)$ for exactly <u>linear non-dissipative</u> vibrations and waves the condition (25) becomes[15]

$$\mathcal{L} = 0 \tag{26}$$

This is trivial since $L = T-U$ and the virial theorem states for periodic motion that $< T > = < U >$ so that the dispersion relation for linear motions, i.e. Equation (26) follows from the virial theorem.

In the dissipative case a principle analogous to the Whitham principle (23) may be found by averaging (16). A proof of the result will be given in the next chapter. We first rewrite (16) using

$$\phi = \phi(\Theta) \tag{27}$$

Substituting into (16) we get

$$\frac{\partial}{\partial x}\left(\frac{d\Theta}{d\phi}\frac{\partial L}{\partial \Theta_x}\right) + \frac{\partial}{\partial t}\left(\frac{d\Theta}{d\phi}\frac{\partial L}{\partial \Theta_t}\right) - \frac{\partial L}{\partial \Theta}\frac{d\Theta}{d\phi} = g\frac{d\phi}{d\Theta} + N - \frac{d\phi}{d\Theta}\omega G \tag{28}$$

Averaging now Equation (28) with $\frac{1}{2\pi}\int . . . d\phi$ and using (24) gives

$$\frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial \Theta_x} + \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \Theta_t} = \frac{g\omega}{2\pi}\int_0^{\pi}\frac{d\phi}{d\Theta}\,d\phi + \frac{1}{2\pi}\int_0^{2\pi} N d\phi - \frac{\omega}{2\pi}\int_0^{2\pi} G\frac{d\phi}{d\Theta}\,d\phi \tag{29}$$

This is the variational principle which replaces (23) in the dissipative case. (13) remains unchanged and an equation replacing (25) will be derived in the next chapter in the course of proving (29).

PERTURBATION APPROACH

As mentioned earlier all equations necessary to determine $k(x, t)$, $\omega(x, t)$ and $A(x, t)$ of (18) can be derived [4,5] using a perturbation technique and without using a Lagrangian. We now extend this perturbation technique to the dissipative case. We prefer this technique because Whitham's method gives wrong results in special cases. If one considers e.g. (7), (15) for $V = N = G = g = 0$, then (25) or (26) do not give the right dispersion relation which is given by (19). Whitham thinks[7] that

this restriction of the form of the Lagrangian presumably corresponds to the assumption of separability if the Hamilton-Jacobi equation in the classical theory of adiabatic invariants. Following Luke[5] we introduce stretched variables by

$$X = \varepsilon x, \qquad T = \varepsilon t \tag{30}$$

where ε is a small parameter which may give a measure of the wave amplitude.[4] In order to include relatively fast local oscillations (through the dependence on the variable Θ) and to take care of the slow variations of A, k and ω (through the dependence on the stretched variables X an. T) we make the expansions

$$\phi(x, t) = U(\Theta, X, T) + \varepsilon U_1(\Theta, X, T) \tag{31}$$

$$V'(\phi) = V'(U) + \varepsilon U_1 V''(U) \tag{32}$$

$$N(\phi_t) = N(-U_\Theta\omega) + \varepsilon(U_T - U_{1\Theta}\omega)N'(-U_\Theta\omega) \tag{33}$$

$$G(\phi) = G(U) + \varepsilon U_1 G'(U) \tag{34}$$

Substituting (31) to (34) into (7) we obtain by equating the various powers of ε (and neglecting ε^2 ...)

$$U_{\Theta\Theta}\left(\frac{\omega^2}{c^2} - k^2\right) + bU - g\omega U_\Theta = -V'(U) + N(-U_\Theta\omega) - G(U)U_\Theta\omega \tag{35}$$

which replaces (25) and

$$U_{1\Theta\Theta}\left(\frac{\omega^2}{c^2} - k^2\right) + bU_1 - g\omega U_{1\Theta} + U_1V''(U) + U_{1\Theta}\omega N' + U_\Theta\omega U_1 G' + U_{1\Theta}\omega G = F \tag{36}$$

where

$$F = \frac{2}{c^2} U_{\Theta T}\omega + 2U_{\Theta X}k + \frac{1}{c^2}U_\Theta\omega_T + U_\Theta k_X - gU_T +$$

$$U_T N' + G(U)U_T \tag{37}$$

The dispersion relation is contained in (35). In order to show that (36) is equivalent to (29) we first show that $U_1 = U_\Theta$ is a solution of the homogeneous equation (37). This is done by derivating (35) with respect to Θ and showing that the result is identical to (37) for $F = 0$. In order to solve (36) we then write

$$U_1 = W(\Theta)U_\Theta \tag{38}$$

substituting into (36) one receives

$$(\frac{\omega^2}{c^2} - k^2)[W_{\Theta\Theta}U_\Theta + 2W_\Theta U_{\Theta\Theta}] - g\omega W_\Theta U_\Theta + \omega N' W_\Theta U_\Theta +$$

$$GW_\Theta U_\Theta = F \tag{39}$$

Using the identity

$$\frac{\partial}{\partial\Theta}(U_\Theta{}^2 W_\Theta) = W_{\Theta\Theta}U_\Theta{}^2 + 2W_\Theta U_{\Theta\Theta}U_\Theta \tag{40}$$

and multiplying (39) by U_Θ we obtain after elimination of w by (38) and integrating

$$(\frac{\omega^2}{c^2} - k^2)\ (U_{1\Theta}U_\Theta - U_1U_{\Theta\Theta}) = \omega \int (U_{1\Theta}U_\Theta - U_1U_{\Theta\Theta})(g - N' - G)d\Theta$$

$$+ \int FU_\Theta d\Theta \tag{41}$$

In order to avoid[17] secular terms proportional to integer powers of Θ U_1 and $U_{1\Theta}$ must be bounded. But U_1 or $U_{1\Theta}$ are not periodic and are unbounded unless the integrals in (41) are bounded for large Θ. Now, if U(and U_Θ,

$U_T = -\omega U_\Theta$) are periodic, then the integrals are bounded. For weak non-linearity and weak linear damping we may assume that U is periodic. Since any nonperiodic function multiplied by a periodic function is itself a periodic function and since integrals over periodic functions over one period are bounded we have from (41) the condition

$$\int F U_\Theta d_\Theta = 0 \tag{42}$$

or using (37)

$$\frac{\partial}{\partial T}\left(\frac{\omega}{c^2}\int_0^{2\pi} U_\Theta^2 d\Theta\right) + \frac{\partial}{\partial X}\left(k\int_0^{2\pi} U_\Theta{}^2\, d\Theta\right) = \int_0^{2\pi} (g\omega - \omega G) U_\Theta dU + \int_0^{2\pi} N dU \tag{43}$$

From (15) and (24) we see that this is equivalent to (29). So now (13), (35) - instead of (25) - and (43) determine $\omega(x, t)$, $k(x, t)$, $A(x, t)$.

APPLICATIONS

In some special cases the non-linear equations can be solved exactly. In these cases $\phi(\Theta)$ is known and (25) can be used. So Luke[5] gives the dispersion relation for the case $N = G = g = b = 0$. This dispersion relation is derived from (25) and also from (35). Another case which can be integrated exactly is $b = V' = G = 0$. Substituting $U_\Theta = W$ in (35) and integration yields

$$\left(\frac{\omega^2}{c^2} - k^2\right)\frac{1}{2\pi}\int_0^{2\pi}\frac{dW}{g\omega W + N} = \Theta - \Theta_o \tag{44}$$

For boundness we have $\Theta - \Theta_o = 1$ and (44) gives a dispersion relation. If, however, $\phi(\Theta)$ is unknown, $\mathcal{L}$ cannot be calculated and (29) cannot be used.

In these cases either an expansion of $\mathcal{L}$ can be used, e.g. for stability problems[18] or the ordinary differential equations (35) and (36) have to be solved, e.g. by the Krylov-Bogolyubov method[3, 17] or by the Lie series method.[19]

I. Nonlinear conservative equation (N = g = G = 0

From (7) and (27), or from (35), we have

$$U_{\Theta\Theta}\left(\frac{\omega^2}{c^2} - k^2\right) + bU + V'(U) = 0 \tag{45}$$

An exact solution (by multiplication by U_Θ) gives from (24), (25) the dispersion relation (amplitude dispersion)

$$\mathcal{L}_E = \sqrt{\frac{\omega^2}{c^2} - k^2} \int_0^{2\pi} \frac{dU}{\sqrt{2(E - V) - bU^2}} - 2\pi = 0 \tag{46}$$

where E is the energy integration constant. So $A = \sqrt{2E}$ = const. The functions $\omega(x, t)$ and $k(x, t)$ can be obtained from (13) and (43). The result A = const for conservative equations may also be derived by the Krylov-Bogolyubov method. For V' = 0, (45) has the generating solution

$$A \sin(\alpha + \psi), \qquad U_\Theta = A\alpha\cos(\alpha\Theta + \psi) \tag{47}$$

where A, α, ψ are constants and

$$\frac{b}{\alpha^2} = \frac{\omega^2}{c^2} - k^2 \tag{48}$$

Now, according to Krylov-Bogolyubov we let $A \to A(\Theta)$, $\psi \to \Psi(\Theta)$ and we then have from (47)

$$A\alpha\cos(\alpha\Theta+\psi) = \dot{A}\sin(\alpha\Theta+\psi) + A(\alpha+\dot{\psi})\cos(\alpha\Theta+\psi) \tag{49}$$

and therefore $(\dot{A} = \frac{dA}{d\Theta})$

$$\dot{A}\sin(\alpha\Theta+\psi) + A\dot{\phi}\cos(\alpha\Theta+\psi) = 0 \tag{50}$$

Substituting $U_{\Theta\Theta} = \dot{A}\alpha\cos(\alpha\Theta+\psi) - A\alpha(\alpha+\dot{\psi})\sin(\alpha\Theta+\psi)$ into (45) gives

$$\dot{A} = -\frac{\alpha}{b} V'\cos(\alpha\Theta+\psi) \tag{51}$$

$$\dot{\psi} = \frac{\alpha}{Ab} V'\sin(\alpha\Theta+\psi) \tag{52}$$

Extending $V'(U)\ \sin(\alpha\Theta+\psi)$ resp $V'(U)\ \cos(\alpha\Theta+\psi)$ into Fourier series and integrating $\frac{1}{2\pi}\int_0^{2\pi} \ldots d\psi$ gives

$$< \dot{A} > = \frac{\alpha}{b} - \frac{1}{2\pi}\int_0^{2\pi} V'(U)\cos\psi d\psi \tag{53}$$

which might be compared to (3) and

$$< \dot{\psi} > = \frac{\alpha}{Ab}\frac{1}{2\pi}\int_0^{2\pi} V'(U)\sin\psi d\psi \tag{54}$$

Using (47), (53) can be written $\int V'(A\sin\psi)d\sin\psi = 0$, and we then have <u>THEOREM I</u>: For any non-linear conservative wave equation of the type (7) the amplitude A (and therefore the energy E) is constant.

Defining an effective frequency

$$\Omega = \alpha + \dot{\psi} \tag{55}$$

we may write (47) in the form

$$U = A(\Theta)\sin(\int\Omega\ d\Theta + \psi_o) \tag{56}$$

where ψ_o = const. Equation (54) and

$$\Theta(x, t) = k(x, t)x - \omega(x, t)t \tag{57}$$

then give a dispersion relation for Ω, see however (46).

II. Non-linear Dissipative Case ($V' = 0$)

From (35) we now have

$$U_{\Theta\Theta}\left(\frac{\omega^2}{c^2} - k^2\right) + bU - g\omega U = N(-U_\Theta\omega) - G(U)U_\Theta\omega \qquad (58)$$

so that V' in (53) and (54) has to be replaced by

$$V' \rightarrow -g\omega A(\Theta)\alpha\cos(\alpha\Theta+\psi) + \omega G(A\sin[\alpha\Theta+\psi]) + N(-\omega A\alpha\cos[\alpha\Theta+\psi])$$

Then from (54) we have $<\dot{\psi}> = 0$ and the THEOREM II: For any non-linear dissipative wave equation of the of the type (7) the frequency ω is not modified by the dissipation terms.

III. Other Problems

The case $b = 0$ presents problems here, since in this case the Krylov-Bogolyubov method cannot be applied to (35). Another interesting problem is

$$-\mu_t + \mu_{xx} = b\mu + \lambda f(\mu) \qquad (59)$$

This equation is of interest in turbulence theory. It can be treated by the methods discussed here.

STABILITY

In many cases the equation (35) cannot be solved exactly or the wave equation is not of the type (7). In these cases Lighthill[18] proposed for conservative systems an expansion of $\mathcal{L}$. We would like to follow up a similar way for the dissipative case. In general, $\mathcal{L} = \mathcal{L}(\omega, k, A)$. We are, however, able to eliminate, e.g., A from one of the three equations determining ω, k, A or from the amplitude dispersion relation. With $\mathcal{L} = \mathcal{L}(\omega, k)$ we rewrite

(29) by exchanging the order of differentiation

$$\Theta_{tt}\mathcal{L}_{\omega\omega} - 2\Theta_{xt}\mathcal{L}_{\omega k} + \Theta_{xx}\mathcal{L}_{kk} = 2\pi Q \tag{60}$$

This is a partial differential equation for $\Theta(x, t)$. We now expand $\mathcal{L}$. Since $\mathcal{L}_{lin} = 0$ according to (36), $\mathcal{L}$ measures exactly the derivation from the dispersion relation of the linear equation. In order to make this more explicit we expand

$$\mathcal{L}(\omega, k) \equiv \mathcal{L}(\tau, k) = a\tau^2 = a(\omega - f[k])^2 \tag{61}$$

where $\omega = f(k)$ is the dispersion relation of the linear equation, e.g. (21). We then have the correspondence $\frac{\partial}{\partial k} \rightarrow \frac{\partial}{\partial k} - f'\frac{\partial}{\partial \tau}$, $\frac{\partial}{\partial \omega} \rightarrow \frac{\partial}{\partial \tau}$ where τ and k are now two independent variables. Since then $\mathcal{L}_k = 0, \mathcal{L}_\tau = 2a\tau, \mathcal{L}_{\tau\tau} = 2a$, (60) takes the form

$$\Theta_{tt} + 2f'\Theta_{xt} + (f'^2 - f''\tau)\Theta_{xx} = \pi Q/a \tag{62}$$

Since $f = f(k)$, $\Theta_x = k(x, t)$. This is a quasilinear partial differential equation for $\Theta(x, t)$. Applying usual characterics method, see e.g. Ref. 20, one may write (62) in the form

$$A\Theta_{xx} + B\Theta_{xt} + C\Theta_{tt} = \pi Q/a \tag{63}$$

Then the characteristics are:

<u>Real</u> - (hyperbolic equation), if $B^2 - 4\,AC \equiv 4f''\tau > 0$

<u>Complex</u> (elliptic equation), if $B^2 - 4\,AC \equiv 4f''\tau < 0$

We see that the dissipative term Q does not enter into this condition. So we have the following <u>THEOREM III</u>:

a. The effect of non-linear terms on the stability behavior is described by $f''(k).(\omega - f[k])$. The stability behavior of the linear equation, described by $\omega = f(k)$ is not altered by non-linear terms, <u>if</u> $f''(k).(\omega - f[k]) > 0$. If however $f''(k).(\omega - f[k]) < 0$, then the non-linear terms may destabilize an otherwise stable solution of a linear equation.

b. The inclusion of dissipative term Q does not by itself modify the character of the stability behavior, but the time behavior of unstable and stable modes is modified.

Examples and applications of this theorm will be given in forthcoming papers.

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